NAME ____________________________

GOSFORD HIGH SCHOOL

2016

Higher School Certificate

MATHEMATICS

Extension 1

Assessment Task 1

General Instructions

- Reading time 5 minutes
- Working time 60 minutes
- Write using black or blue pen
- Board approved calculators may be used
- For all questions show relevant mathematical reasoning and/or calculations
- Start each question on a new page
- Total marks 42

Question 1 Mathematical Induction (10 marks)

a) Prove, using mathematical induction, that

$5^n + 3$ is divisible by 4 , for all integers $n \geq 1$ (4)

b) Prove, by mathematical induction, that

$$1 \times 2^0 + 2 \times 2^1 + 3 \times 2^2 + \cdots\cdots + n \times 2^{n-1} = 1 + (n-1) \times 2^n \quad (4)$$

c) **Consider** the statement that

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

is true for all positive integral values of n.

Prove that the result is true for $n = 3$. (2)

(NOTE that you are NOT asked to PROVE the result in (c) by MATHEMATICAL INDUCTION)

Question 2 Polynomials (16 marks)

a) (i) Find the remainder when $2x^3 - 5x^2 - 4x + 3$ is divided by $(x + 2)$ (1)

(ii) Determine the value of p for which $(x - p)$ is a factor of $A(x)$, given $A(x) = x^3 + px^2 - (2p^2 + 12)x + (7p + 10)$ (2)

b) Neatly sketch $y = (1 - x)^3(x + 2)$ without the use of calculus, (*do not attempt to find the turning points*) (2)

c) If α, β, and γ are the roots of the equation $x^3 - 2x^2 - 3x + 4 = 0$, determine the value of :

(i) $\alpha + \beta + \gamma$ (ii) $\alpha\beta + \alpha\gamma + \beta\gamma$ (iii) $\alpha^2 + \beta^2 + \gamma^2$ (4)

d) Find the value of the constant m in the equation $4x^3 + 32x^2 + mx + 60 = 0$, given that one root is equal to the sum of the other two. (3)

e) The polynomial $(x - a)^3 + b$ is equal to zero at $x = 1$ and when divided by x, the remainder is -7.

Find all possible values of a and b. (4)

Question 3 Parametric Representation (16 marks)

a) Two distinct points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$,

(i) show that the gradient of the tangent at P is p (1)

(ii) show that the equation of the tangent to the parabola at P is $px - y - ap^2 = 0$ (1)

(iii) show that equation of the chord PQ is $(p + q)x - 2y - 2apq = 0$ (2)

b) A parabola has parametric equations $x = 4t,\ y = -2t^2$.

(i) Find the cartesian equation of this parabola. (1)

(ii) The normal at any point $(4t, -2t^2)$, where $t \neq 0$, on the parabola cuts the y-axis at G.
Find the equation of this normal and the coordinates of G. (3)

(iii) Find the cartesian equation of the locus of the point M, if M is the midpoint of PG. (2)

c) The point $P(2ap, ap^2)$ lies on the parabola $x^2 = 4ay$, where $a > 0$.

The equation of the normal at P is $x + py = 2ap + ap^3$,
NOTE you do not need to prove this.

(i) This normal meets the parabola again at the point $Q(2aq, aq^2)$.
Prove that $p^2 + pq + 2 = 0$ (3)

(ii) If the chords OP and OQ are perpendicular, where O is the origin, show that $p^2 = 2$. (3)

GHS 2016 HSC MATHEMATICS EXTENSION 1

ASSESSMENT TASK 1

SOLUTIONS

Question 1

a) Prove true for $n=1$

$5^n + 3 = 5^1 + 3$ when $n = 1$

$= 8$ which is divisible by 4

$\therefore$ true for $n = 1$

Assume true for $n = k$

i.e. Assume $5^k + 3 = 4M$ where $M \in J^+$

$\therefore 5^k = 4M - 3$

Prove true for $n = k+1$, if true for $n = k$

i.e. Prove that $5^{k+1} + 3$ is divisible by 4

Now $5^{k+1} + 3 = 5 \times 5^k + 3$

$= 5 \times (4M - 3) + 3$ (using assumption)

$= 20M - 15 + 3$

$= 20M - 12$

$= 4(5M - 3)$

which is divisible by 4 since $(5M-3)$ is a positive integer.

$\therefore$ true for $n = k+1$, if true for $n = k$.

Since true for $n = 1$ and true for $n = k+1$ if true for $n = k$ therefore by mathematical induction statement is true for all given 'n'.

b) Prove true for $n = 1$

L.H.S. $= 1 \times 2^{1-1}$

$= 1 \times 2^0$

$= 1$

L.H.S. $= 1 + (1-1) \times 2^1$

$= 1$

$=$ R.H.S.

$\therefore$ true for $n = 1$

Assume true for $n = k$

i.e. $1 \times 2^0 + 2 \times 2^1 + \dots + k \times 2^{k-1} = 1 + (k-1) \times 2^k$

Prove true for $n = k+1$, if true for $n = k$

i.e $1 \times 2^0 + 2 \times 2^1 + \dots + k \times 2^{k-1} + (k+1) \times 2^k = 1 + k \times 2^{k+1}$

L.H.S. $= 1 + (k-1) \times 2^k + (k+1) \times 2^k$

$= 1 + 2^k[(k-1) + (k+1)]$

$= 1 + 2^k \times 2k$

$= 1 + k \times 2^{k+1}$

$=$ R.H.S.

$\therefore$ true for $n = k+1$, if true for $n = k$

Since true for $n = 1$, therefore true for $n = 1+1 = 2$

Since true for $n = 2$, therefore true for $n = 2+1 = 3$

and so on $\therefore$ By induction true for all given 'n'.

c) L.H.S. $= \sum_{k=1}^{n} k^3$

$= 1^3 + 2^3 + 3^3$

$= 1 + 8 + 27$

$= 36$

R.H.S. $= \dfrac{3^2(3+1)^2}{4}$

$= \dfrac{9 \times 16}{4}$

$= 36$

$=$ L.H.S.

$\therefore$ true for $n = 3$

Question 2 Polynomials

a) i) Let $P(x) = 2x^3 - 5x^2 - 4x + 3$

Remainder $= P(-2)$

$= 2(-2)^3 - 5(-2)^2 - 4(-2) + 3$

$= -16 - 20 + 8 + 3$

$= -25$

ii) $A(p) = 0$

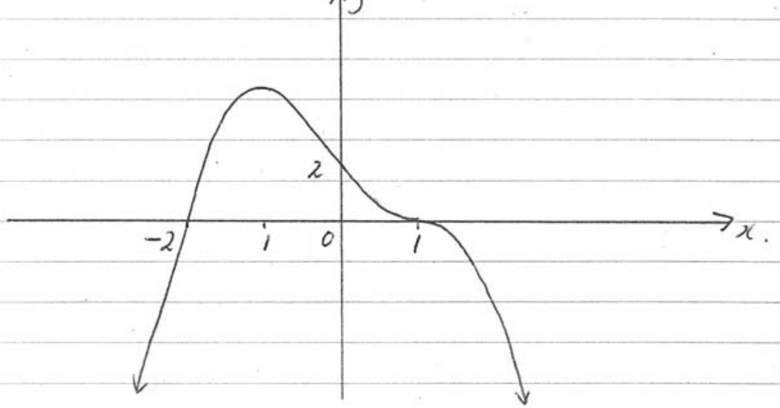
$\therefore (p)^3 + p(p)^2 - (2p^2+12)\cdot(p) + (7p+10) = 0$

$p^3 + p^3 - 2p^3 - 12p + 7p + 10 = 0$

$-5p + 10 = 0$

$p = 2$

b)



c) (i) $\alpha + \beta + \gamma = \dfrac{-b}{a}$

$= 2$

(ii) $\alpha\beta + \alpha\gamma + \beta\gamma = \dfrac{c}{a}$

$= -3$

(iii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= 2^2 - 2(-3)$

$= 10$

d) Let the roots be α, β and $\alpha + \beta$

$\therefore \alpha + \beta + (\alpha + \beta) = \dfrac{-32}{4}$

$2(\alpha + \beta) = -8$

$\alpha + \beta = -4$

But $(\alpha + \beta)$ is a root $\longrightarrow -4$ is a root (zero)

$\therefore 4(-4)^3 + 32(-4)^2 + m(-4) + 60 = 0$

$-256 + 256 - 4m + 60 = 0$

$4m = 60$

$m = 15$

e) Let $P(x) = (x-a)^3 + b$

$P(1) = 0$

$(1-a)^3 + b = 0$

$P(0) = -7$

$(0-a)^3 + b = -7$

$b = a^3 - 7$

Solving simultaneously

$(1-a)^3 + a^3 - 7 = 0$

$\therefore 1 - 3a + 3a^2 - a^3 + a^3 - 7 = 0$

$3a^2 - 3a - 6 = 0$

$a^2 - a - 2 = 0$

$(a-2)(a+1) = 0$

$a = 2, -1$

when $a = 2$, $b = 1$

when $a = -1$, $b = -8$

Question 3

a) (i)

$$x^2 = 4ay$$
$$y = \frac{x^2}{4a}$$

$$\frac{dy}{dx} = \frac{2x}{4a}$$
$$= \frac{x}{2a}$$
$$= \frac{2ap}{2a} \quad \text{at } P$$
$$= p$$

$\therefore$ Gradient of Tangent at P is p.

(ii) Equation of Tangent is $y - ap^2 = p(x - 2ap)$

ie $y - ap^2 = px - 2ap^2$

$\therefore px - y - ap^2 = 0$

(iii) Gradient of chord $PQ = \dfrac{ap^2 - aq^2}{2ap - 2aq}$

$$= \frac{a(p-q)(p+q)}{2a(p-q)}$$
$$= \frac{p+q}{2}$$

Equation of Chord PQ is $y - ap^2 = \left(\frac{p+q}{2}\right)(x - 2ap)$

ie $2y - 2ap^2 = (p+q)x - 2ap(p+q)$

$$2y - 2ap^2 = (p+q)x - 2ap^2 - 2apq.$$

$$(p+q)x - 2y - 2apq = 0$$

b) (i) $x = 4t$, $y = -2t^2$

$t = \frac{x}{4} \longrightarrow y = -2\left(\frac{x}{4}\right)^2$

$$y = \frac{-2x^2}{16}$$
$$y = \frac{-x^2}{8}$$

or $x^2 = -8y$.

(ii) $y = \frac{-x^2}{8} \longrightarrow \frac{dy}{dx} = \frac{-2x}{8}$

$$= \frac{-x}{4}.$$
$$= \frac{-4t}{4} \quad \text{at } x = 4t$$
$$= -t$$

$\therefore$ Normal has gradient $\frac{1}{t}$

$\therefore$ Equation of Normal is $y + 2t^2 = \frac{1}{t}(x - 4t)$

ie $ty + 2t^3 = x - 4t$

$$x - ty = 4t + 2t^3$$

at G, $x = 0$ $\quad \therefore -ty = 4t + 2t^3$

$\therefore \quad y = -4 - 2t^2$

$\therefore$ G has coordinates $\left(0, -4-2t^2\right)$

M has coordinates $\left[\dfrac{4t}{2}, \dfrac{-4-2t^2-2t^2}{2}\right]$

$= \left[2t, \dfrac{-4-4t^2}{2}\right]$

$= \left[2t, -2-2t^2\right]$

$\therefore$ Parametric Equations of M are

$x = 2t \quad , \quad y = -2-2t^2$

$t = \dfrac{x}{2} \longrightarrow y = -2 - 2\left(\dfrac{x}{2}\right)^2$

$y = -2 - \dfrac{x^2}{2}$

$2y = -4 - x^2$

$\therefore$ Locus of M is $\quad x^2 = -2y - 4$

c) (i) Normal has equation $x + py = 2ap + ap^3$

$Q(2aq, aq^2)$ satisfies

$\therefore \quad 2aq + p \cdot aq^2 = 2ap + ap^3$

$2q + pq^2 = 2p + p^3$

$2q - 2p = p^3 - pq^2$

$-2(p-q) = p(p^2-q^2)$

$-2(p-q) = p(p-q)(p+q)$

$-2 = p(p+q) \quad$ since $p \neq q$.

$-2 = p^2 + pq$

$\therefore \quad p^2 + pq + 2 = 0$

(ii) Gradient $(M_{OP}) = \dfrac{ap^2}{2ap}$ $\quad$ Gradient $(M_{OQ}) = \dfrac{q}{2}$

$= \dfrac{p}{2}$

$M_{OP} \times M_{OQ} = -1.$

$\therefore \dfrac{p}{2} \times \dfrac{q}{2} = -1$

$\therefore pq = -4$

$\therefore$ using (i) $\quad p^2 - 4 + 2 = 0$

$\therefore p^2 = 2$